

NUMBER OF POINTS ON THE FULL MODULI SPACE OF CURVES OVER FINITE FIELDS

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ABSTRACT. The distribution of the number of points on abelian covers of $\mathbb{P}^1(\mathbb{F}_q)$ ranging over an irreducible moduli space has been answered in a recent work by the author [10],[11]. The authors of [1] determined the distribution over the whole moduli space for curves with $\text{Gal}(K(C)/K)$ a prime cyclic. In this paper, we prove a result towards determining the distribution over the whole moduli space of curves with $\text{Gal}(K(C)/K)$ any abelian group. We successfully determine the distribution in the case $\text{Gal}(K(C)/K)$ is a power of a prime cyclic.

1. INTRODUCTION

Let \mathcal{H} be a family of smooth, projective curves over \mathbb{F}_q , the finite field with q elements. We are interested in determining the probability that a curve, chosen randomly from our family, has a given number of points. Classical results due to Katz and Sarnak [7] tell us what happens if we fix the genus of the curve, g and let $q \rightarrow \infty$. Progress has been made in the other situation when q is fixed and $g \rightarrow \infty$.

Let $K = \mathbb{F}_q(X)$ and $K(C)$ be the field of functions of C , a curve over \mathbb{F}_q . Then $K(C)$ will be a finite field extension of K . Moreover, if we fix a copy of $\mathbb{P}^1(\mathbb{F}_q)$, then every such finite extension corresponds to a smooth, projective curve (Corollary 6.6 and Theorem 6.9 from Chapter I of [6]). If $K(C)$ is a Galois extension of K , denote $\text{Gal}(C) := \text{Gal}(K(C)/K)$ and $g(C)$ to be the genus of C . Define

$$\mathcal{H}_{G,g} = \{C : \text{Gal}(C) = G, g(C) = g\}.$$

We want to determine the probability, that a random curve in this family has a given number of points. That is, for every $N \in \mathbb{Z}_{\geq 0}$, we want to determine

$$\text{Prob}(C \in \mathcal{H}_{G,g} : \#C(\mathbb{P}^1(\mathbb{F}_q)) = N) := \frac{|\{C \in \mathcal{H}_{G,g} : \#C(\mathbb{P}^1(\mathbb{F}_q)) = N\}|}{|\mathcal{H}_{G,g}|}.$$

Therefore, the first thing we need to do is determine $|\mathcal{H}_{G,g}|$. Wright [13] was the first to answer such a question. He proved that if G is abelian and $q \equiv 1 \pmod{\exp(G)}$, (where $\exp(G)$ is the smallest integer such that $ng = e$ for all $g \in G$) then as $g \rightarrow \infty$

$$(1.1) \quad \sum_{j=0}^{N-1} q^{-\frac{j}{N}} |\mathcal{H}_{G,g+j}| \sim C(K, G) g^{\phi_G(Q)-1} q^{\frac{g}{N}}$$

where $C(K, G)$ is a non-zero constant, $N = |G| - \frac{|G|}{Q}$ where Q is the smallest prime divisor of $|G|$ and $\phi_G(Q)$ is the number of elements of G of order Q . (Note: Wright's result does not require $q \equiv 1 \pmod{\exp(G)}$, but we will always assume that here and it makes the formula slightly nicer).

Remark 1.1. From now on the function $\phi_G(s)$ will be the number of elements of G of order s .

Bucur, David, Feigon, Kaplan, Lalin, Ozman and Wood [1] shows that if Q is a prime then as $g \rightarrow \infty$

$$|\mathcal{H}_{\mathbb{Z}/Q\mathbb{Z},g}| = \begin{cases} c_{Q-2} q^{\frac{2g+2Q-2}{Q-1}} P\left(\frac{2g+2Q-2}{Q-1}\right) + O\left(q^{(\frac{1}{2}+\epsilon)\frac{2g+2Q-2}{Q-1}}\right) & g \equiv 0 \pmod{Q-1} \\ 0 & \text{otherwise} \end{cases}$$

where c_{Q-2} is a constant that they make explicit and P is a monic polynomial of degree $Q-2$.

Our first result is extending this to any abelian group. But first, we must define a quasi-polynomial.

Definition 1.2. A quasi-polynomial is a function that can be written as

$$p(x) = c_n(x)x^n + c_{n-1}(x)x^{n-1} + \cdots + c_0(x)$$

where $c_i(x)$ is a periodic function with integer period. We call the c_i the coefficients of the quasi-polynomial. Moreover, if $c_n(x)$ is not identically the zero function then we say p has degree n and call it the leading coefficient.

Definition 1.3. Let $\mathcal{R} = [0, \dots, r_1 - 1] \times \cdots \times [0, \dots, r_n - 1] \setminus \{(0, \dots, 0)\}$ be a set of inter-valued vectors. For any $\vec{\alpha} \in \mathcal{R}$ let

$$e(\vec{\alpha}) = \text{lcm}_{j=1, \dots, n} \left(\frac{r_j}{(r_j, \alpha_j)} \right)$$

Theorem 1.4. Let G be any abelian group and $q \equiv 1 \pmod{\exp(G)}$. If there exists some $(d(\vec{\alpha}))_{\vec{\alpha} \in \mathcal{R}}$ such that

$$2g + 2|G| - 2 = \sum_{\vec{\alpha} \in \mathcal{R}} \left(|G| - \frac{|G|}{e(\vec{\alpha})} \right) d(\vec{\alpha}) + |G| - \frac{|G|}{e(\vec{d})}$$

where $\vec{d} = (d_1, \dots, d_n)$ and

$$d_j = \sum_{\vec{\alpha} \in \mathcal{R}} \alpha_j d(\vec{\alpha})$$

then

$$|\mathcal{H}_{G,g}| = \sum_{j=1}^{\eta} P_j(2g) q^{\frac{2g+2|G|-2}{|G|-\frac{|G|}{s_j}}} + O\left(q^{\frac{(1+\epsilon)g}{|G|-\frac{|G|}{s_1}}}\right)$$

where $1 = s_0 < s_1 < \cdots < s_{\eta} = \exp(G)$ are the divisors of $\exp(G)$, P_1 is a quasi-polynomial of degree $\phi_G(s_1) - 1$ and P_j is a quasi-polynomial of degree at most $\phi_G(s_j) - 1$. If no such solution exists then $|\mathcal{H}_{G,g}| = 0$.

If we restrict to the case $G = (\mathbb{Z}/Q\mathbb{Z})^n$, then we can say more about the polynomials.

Corollary 1.5. If $G = (\mathbb{Z}/Q\mathbb{Z})^n$ for Q a prime, $q \equiv 1 \pmod{Q}$, and $2g + 2Q^n - 2 \equiv 0 \pmod{Q^n - Q^{n-1}}$, then

$$|\mathcal{H}_{G,g}| = P\left(\frac{2g + 2Q^n - 2}{Q^n - Q^{n-1}}\right) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}} + O\left(q^{\frac{(1+\epsilon)g}{Q^n-Q^{n-1}}}\right)$$

where P is a polynomial of degree $Q^n - 2$ with leading coefficient

$$\frac{1}{(Q^n - 2)!} \frac{q + Q^n - 1}{q} \frac{L_{Q^n-2}}{\zeta_q(2)^{Q^n-1}}$$

where L_{Q^n-2} is a constant defined in (9.2) and $\zeta_q(s)$ is the zeta function for $\mathbb{F}_q[X]$ ((9.3)). If $2g + 2Q^n - 2 \not\equiv 0 \pmod{Q^n - Q^{n-1}}$, then $|\mathcal{H}_{G,g}| = 0$

If $q \equiv 1 \pmod{\exp(G)}$ and G is abelian, then we can write

$$\mathcal{H}_{G,g} = \bigcup \mathcal{H}^{\vec{d}(\vec{\alpha})}$$

where $\mathcal{H}^{\vec{d}(\vec{\alpha})}$ is an irreducible moduli space of $\mathcal{H}_{G,g}$ (see Section 2 of [10] for a full treatment of this).

For specific classes of groups, several authors determined that as $d(\vec{\alpha}) \rightarrow \infty$ for all $\vec{\alpha}$, then

$$(1.2) \quad \text{Prob}(C \in \mathcal{H}^{\vec{d}(\vec{\alpha})} : \#C(\mathbb{P}^1(\mathbb{F}_q)) = N) \sim \text{Prob}\left(\sum_{i=1}^{q+1} X_i = N\right)$$

where the X_i are i.i.d. random variables that can be made completely explicit.

Kurlberg and Rudnick [8] were the first to do this for hyper-elliptic curves ($G = \mathbb{Z}/2\mathbb{Z}$). Bucur, David, Feigon and Lalin [2],[3] then extended this to prime cyclic curves ($G = \mathbb{Z}/Q\mathbb{Z}$, Q a prime). Lorenzo, Meleleo and Milione [9] then proved this for n -quadratic curves ($G = (\mathbb{Z}/2\mathbb{Z})^n$). The author [10],[11] completes this for any abelian group.

The fact that these results need all the $d(\vec{\alpha}) \rightarrow \infty$ is why we can not deduce the results for the whole space from the results for the subspaces. However, in the case $G = (\mathbb{Z}/Q\mathbb{Z})^n$, we can deduce the main term of Corollary 1.5 from these results. However, the error term we get from doing this is just $(1 + o(1))$. Likewise we can do the same for Corollary 1.7 and Theorem 1.8.

If $G = \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_n\mathbb{Z}$, then since we assume $q \equiv 1 \pmod{\exp(G)}$, by Kummer Theory, we can find $F_j \in \mathbb{F}_q[X]$, r_j^{th} -power free such that

$$K(C) = K\left({}^r\sqrt{F_1(X)}, \dots, {}^r\sqrt{F_n(X)}\right).$$

Fix an ordering x_1, \dots, x_{q+1} of $\mathbb{P}^1(\mathbb{F}_q)$ such that x_{q+1} is the point at infinity. This ordering will be fixed for the rest of this paper. Then, the number of points on the curve depend on the values of

$$\chi_{r_j}(F_j(x_i)), j = 1, \dots, n, i = 1, \dots, q+1$$

where $\chi_{r_j} : \mathbb{F}_q \rightarrow \mu_{r_j}$ is a multiplicative character of order r_j . Moreover, the value of $F_j(x_{q+1})$ depends on the leading coefficient and degree of $F_j(X)$. (Again, see [11] for more on this.) Therefore, we will define

$$\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$$

$$E = \begin{pmatrix} \epsilon_{1,1} & \cdots & \epsilon_{1,n} \\ \vdots & \ddots & \vdots \\ \epsilon_{\ell,1} & \cdots & \epsilon_{\ell,n} \end{pmatrix}$$

such that $0 \leq \ell \leq q$, $\epsilon_{i,j} \in \mu_{r_j}$. Define

$$\mathcal{H}_{G,g}(\vec{k}, E) = \{C \in \mathcal{H}_{G,g} : \deg(F_j) \equiv k_j \pmod{r_j}, \chi_{r_j}(F_j(x_i)) = \epsilon_{i,j}, \\ i = 1, \dots, \ell, j = 1, \dots, n\}.$$

Then we get

Theorem 1.6. *Let G be any abelian group and $q \equiv 1 \pmod{\exp(G)}$. If there exists some $(d(\vec{\alpha}))_{\vec{\alpha} \in \mathcal{R}}$ such that*

$$2g + 2|G| - 2 = \sum_{\vec{\alpha} \in \mathcal{R}} \left(|G| - \frac{|G|}{e(\vec{\alpha})} \right) d(\vec{\alpha}) + |G| - \frac{|G|}{e(\vec{k})}$$

where

$$d_j = \sum_{\vec{\alpha} \in \mathcal{R}} \alpha_j d(\vec{\alpha}) \equiv k_j \pmod{r_j}$$

then

$$|\mathcal{H}_{G,g}(\vec{k}, E)| = \sum_{j=1}^{\eta} P_{j;\vec{k},E}(2g) q^{\frac{2g+2|G|-2}{|G|-\frac{|G|}{s_j}}} + O\left(q^{\frac{(1+\epsilon)g}{|G|-\frac{|G|}{s_1}}}\right)$$

where $1 = s_0 < s_1 < \dots < s_{\eta} = r_n$ are the divisors of r_n and $P_{j;\vec{k},E}$ is a quasi-polynomial of degree at most $\phi_G(s_j) - 1$. If there no such solution exists then $|\mathcal{H}_{G,g}(\vec{k}, E)| = 0$.

Again, if $G = (\mathbb{Z}/Q\mathbb{Z})^n$, then we can say more

Corollary 1.7. *If $G = (\mathbb{Z}/Q\mathbb{Z})^n$, for Q a prime, $q \equiv 1 \pmod{Q}$ and $2g + 2Q^n - 2 \equiv 0 \pmod{Q^n - Q^{n-1}}$ then*

$$|\mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n,g}(\vec{k}, E)| = P_{\vec{k},E}\left(\frac{2g + 2Q^n - 2}{Q^n - Q^{n-1}}\right) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}} + O\left(q^{\frac{(1+\epsilon)g}{Q^n-Q^{n-1}}}\right)$$

where $P_{\vec{k},E}$ is a polynomial of degree $Q^n - 2$ with leading coefficient

$$\frac{(q-1)^n}{Q^n(Q^n-2)!} \frac{L_{Q^n-2}}{\zeta_q(2)^{Q^n-1}} \left(\frac{q}{Q^n(q+Q^n-1)} \right)^{\ell}.$$

If $2g + 2Q^n - 2 \not\equiv 0 \pmod{Q^n - Q^{n-1}}$ then $|\mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n,g}(\vec{k}, E)| = 0$.

Now, using Corollaries 1.5 and 1.7 we can prove a result on the distribution of the number of points for the whole space.

Theorem 1.8. *Let $G = (\mathbb{Z}/Q\mathbb{Z})^n$ and fix q such that $q \equiv 1 \pmod{Q}$. If $2g + 2Q^n - 2 \equiv 0 \pmod{Q^n - Q^{n-1}}$ then as $g \rightarrow \infty$,*

$$\frac{|\{C \in \mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n,g} : \#C(\mathbb{P}^1(\mathbb{F}_q)) = M\}|}{|\mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n,g}|} = \text{Prob}\left(\sum_{i=1}^{q+1} X_i = M\right) \left(1 + O\left(\frac{1}{g}\right)\right)$$

where the X_i are i.i.d. random variables taking value 0, Q^n or Q^{n-1} such that

$$X_i = \begin{cases} Q^{n-1} & \text{with probability } \frac{Q^n-1}{Q^{n-1}(q+Q^n-1)} \\ Q^n & \text{with probability } \frac{q}{Q^n(q+Q^n-1)} \\ 0 & \text{with probability } \frac{(Q^n-1)(q+Q^n-Q)}{Q^n(q+Q^n-1)} \end{cases}.$$

Remark 1.9. The proof of Theorem 1.8 follows directly from Corollaries 1.5 and 1.7 and the work done in [10] and [11]. Therefore, if we were able to determine the leading coefficients of P_1 in Theorem 1.4 and $P_{1,\vec{k},E}$ in Theorem 1.6, then an analogous result as Theorem 1.8 would follow from the work done in [10] and [11].

Remark 1.10. The random variables appearing in Theorem 1.8 are the same that appear in (1.2) in the case $G = (\mathbb{Z}/Q\mathbb{Z})^n$.

Remark 1.11. Bucur, David, Feigon, Kaplan, Lalin, Ozman and Wood [1] prove analogous results to Corollary 1.7, and Theorem 1.8 for $G = \mathbb{Z}/Q\mathbb{Z}$.

2. NOTATION AND SETUP

From now on, we will assume that $G = \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_n\mathbb{Z}$ such that $r_j | r_{j+1}$ and $q \equiv 1 \pmod{r_n}$. Under these assumptions we can apply Kummer theory (Chap.14 Proposition 37 of [5]) to find $F_j \in \mathbb{F}_q[X]$, r_j^{th} -power free for $j = 1, \dots, n$ such that

$$K(C) = K\left(\sqrt[r_1]{F_1(X)}, \dots, \sqrt[r_n]{F_n(X)}\right).$$

Let

$$\begin{aligned} \mathcal{H}_{G,g}^* &= \{C \in \mathcal{H}_{G,g} : F_j \text{ is monic}\} \\ \mathcal{H}_{G,g}^*(\vec{k}, E) &= \{C \in \mathcal{H}_{G,g}(\vec{k}, E) : F_j \text{ is monic}\}. \end{aligned}$$

We call curves in $\mathcal{H}_{G,g}^*$ *monic*.

Now for each F_j , it's leading coefficient can be chosen from any of the equivalence classes of $\mathbb{F}_q^*/(\mathbb{F}_q^*)^{r_j}$ to give a different extension (and thus curve). Therefore we see that $|\mathcal{H}_{G,g}| = |G| |\mathcal{H}_{G,g}^*|$ and $|\mathcal{H}_{G,g}(\vec{k}, E)| = |G| |\mathcal{H}_{G,g}^*(\vec{k}, E)|$. Therefore, we will work with $\mathcal{H}_{G,g}^*$ and $\mathcal{H}_{G,g}^*(\vec{k}, E)$ from now on.

Now, define

$$\mathcal{R} = [0, \dots, r_1 - 1] \times \cdots \times [0, \dots, r_n - 1] \setminus \{(0, \dots, 0)\}$$

to be set of integer valued vectors denoted as $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ such that not all of them are 0. Denote

$$\mathcal{R}' = \mathcal{R} \cup \{(0, \dots, 0)\}.$$

Then for every $\vec{\alpha} \in \mathcal{R}$ let

$$f_{\vec{\alpha}} = \prod_{\substack{P \\ v_P(F_j) = \alpha_j}} P$$

where the product is over prime polynomials of $\mathbb{F}_q[X]$. Then we can write

$$F_j = \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}}^{\alpha_j}$$

where we use the convention that f^0 is identically the constant polynomial 1. Moreover, all the $f_{\vec{\alpha}}$ are squarefree and pairwise coprime.

In [11], the author uses the Riemann-Hurwitz formula (Theorem 7.16 of [12]) to show that the genus of C satisfies the relation

$$(2.1) \quad 2g + 2|G| - 2 = \sum_{\vec{\alpha}} \left(|G| - \frac{|G|}{e(\vec{\alpha})} \right) \deg(f_{\vec{\alpha}}) + |G| - \frac{|G|}{e(\vec{d})}$$

where $\vec{d} = (d_1, \dots, d_n) = (\deg(F_1), \dots, \deg(F_n))$ and for any vector $\vec{v} = (v_1, \dots, v_n)$,

$$(2.2) \quad e(\vec{v}) := \text{lcm}_{j=1, \dots, n} \left(\frac{r_j}{(r_j, v_j)} \right).$$

Notice, that the genus only depends on the degree of the $f_{\vec{\alpha}}$ and the congruence class of the d_j modulo r_j . Therefore, define

$$(2.3) \quad \mathcal{F}_d = \{f \in \mathbb{F}_q[X] : f \text{ is monic, squarefree and } \deg(f) = d\}$$

$$(2.4) \quad \mathcal{F}_{\vec{d}(\vec{\alpha})} = \{(f_{\vec{\alpha}}) \in \prod_{\vec{\alpha} \in \mathcal{R}} \mathcal{F}_{d(\vec{\alpha})} : (f_{\vec{\alpha}}, f_{\vec{\beta}}) = 1 \text{ for all } \vec{\alpha} \neq \vec{\beta} \in \mathcal{R}\}$$

where $\vec{d}(\vec{\alpha}) = (d(\vec{\alpha}))_{\vec{\alpha} \in \mathcal{R}}$ is a vector of non-negative integers indexed by the vectors of \mathcal{R} .

Now for any $\vec{k} = (k_1, \dots, k_n) \in \mathcal{R}'$ consider the congruence conditions

$$(2.5) \quad \sum_{\vec{\alpha} \in \mathcal{R}} \alpha_j d(\vec{\alpha}) \equiv k_j \pmod{r_j}, j = 1, \dots, n.$$

Further, let E be an $\ell \times n$ matrix such that

$$(2.6) \quad E = \begin{pmatrix} \epsilon_{1,1} & \dots & \epsilon_{1,n} \\ \vdots & \ddots & \vdots \\ \epsilon_{\ell,1} & \dots & \epsilon_{\ell,n} \end{pmatrix}$$

with $\epsilon_{i,j} \in \mu_{r_j}$. Now, define

$$(2.7) \quad \mathcal{F}_{\vec{d}(\vec{\alpha}); \vec{k}, E} = \begin{cases} \{(f_{\vec{\alpha}}) \in \mathcal{F}_{\vec{d}(\vec{\alpha})} : \chi_{r_j}(F_j(x_i)) = \epsilon_{i,j}, i = 1, \dots, \ell, j = 1, \dots, n\} & (2.5) \text{ is satisfied} \\ \emptyset & \text{otherwise} \end{cases}$$

Finally, define

$$(2.8) \quad \mathcal{F}_{D; \vec{k}, E} = \bigcup_{\vec{d}(\vec{\alpha})} \mathcal{F}_{\vec{d}(\vec{\alpha}); \vec{k}, E}$$

where the union is over all $\vec{d}(\vec{\alpha})$ that satisfies

$$(2.9) \quad D = \sum_{\vec{\alpha} \in \mathcal{R}} c(\vec{\alpha}) d(\vec{\alpha})$$

where $c(\vec{\alpha})$ are some fixed constants.

If we set $c(\vec{\alpha}) = |G| - \frac{|G|}{e(\vec{\alpha})}$ and $D = 2g + 2|G| - 2 - c(\vec{k})$, then we see that (2.9) becomes (2.1). Outside of Section 3, we will work with arbitrary $c(\vec{\alpha})$ with the idea that eventually we will set them equal to what we need.

Therefore, it seems as if what we need to do is determine the size of $\mathcal{F}_{D; \vec{k}, E}$ for all D, \vec{k}, E . Unfortunately, this will actually count too many curves! However, there is still a way to determine the size of $\mathcal{H}_{G,g}^*$ and $\mathcal{H}_{G,g}^*(\vec{k}, E)$ if we know $|\mathcal{F}_{D; \vec{k}, E}|$.

3. TOO MANY CURVES

Ideally, we would like to say that every monic curve, C , such that $\text{Gal}(C) = G$, $g(C) = g$, comes from an element $\mathcal{F}_{\vec{d}(\vec{\alpha})}$ such that $\vec{d}(\vec{\alpha})$ satisfies (2.1). Unfortunately, this is not true.

For example, if we consider the set $\mathcal{F}_{(0, d_2, 0)}$ such that $2g + 6 = 2d_2$ and $2d_2 \equiv 0 \pmod{4}$. Then $(0, d_2, 0)$ satisfies (2.1) for $G = \mathbb{Z}/4\mathbb{Z}$ and we would hope that this would correspond to a curve with $\text{Gal}(C) = \mathbb{Z}/4\mathbb{Z}$ and $g(C) = g$. However, an element of $\mathcal{F}_{(0, d_2, 0)}$ would look like $(1, f_2, 1)$ where f_2 is a square-free polynomial of degree d_2 . This would correspond to a curve with affine model $Y^4 = f_2^2$, which clearly has $K(C) = K(\sqrt{f_2})$ and so $\text{Gal}(C) = \mathbb{Z}/2\mathbb{Z}$. Moreover,

$$g(C) = \frac{d_2 - 2}{2} = \frac{g + 3 - 2}{2} = \frac{g - 1}{2} + 1.$$

It is easy to see how this argument can be extended to any group G that does not have prime order. Indeed, what we will show in this section is that the elements of $\mathcal{F}_{\vec{d}(\vec{\alpha})}$ correspond to monic curves whose Galois group is a *subgroup* of G .

Remark 3.1. When we talk about all the subgroups of G , we mean all the different possible subsets of G that are subgroups of G . That is, two subgroups $H, H' \subset G$ are said to be the same subgroup if and only if they are equal as subsets. For example, if $G = \mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$, then the subgroups

$$\{(a, 0) : 0 \leq a \leq Q - 1\}$$

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are all different even though they are all isomorphic to $\mathbb{Z}/Q\mathbb{Z}$.

For simplicity, in this section, we will fix $c(\vec{\alpha}) = |G| - \frac{|G|}{e(\vec{\alpha})}$.

Proposition 3.2. *Let*

$$M(G, g) = \{C, \text{ monic} : \text{Gal}(C) = H \subset G, g(C) = \frac{g-1}{|G|/|H|} + 1\}.$$

Then there is a natural bijection from elements of

$$\bigcup_{\vec{d}(\vec{\alpha})} \mathcal{F}_{\vec{d}(\vec{\alpha})}$$

to $M(G, g)$ where the union is over all $\vec{d}(\vec{\alpha})$ that satisfies (2.1).

Proof. Let $(f_{\vec{\alpha}}) \in \mathcal{F}_{\vec{d}(\vec{\alpha})}$ for some fixed $\vec{d}(\vec{\alpha})$ that satisfies (2.1). Define

$$\mathcal{R}^* = \{\vec{\alpha} \in \mathcal{R} : d(\vec{\alpha}) \neq 0\}.$$

For every $\vec{\alpha} \in \mathcal{R}$, we can identify it as an element in G in the natural way. Let $H \subset G$ be the subgroup that is generated by \mathcal{R}^* under this identification. (From now on, in this proof, we will identify elements of H and G with elements of \mathcal{R}). We will show that $\text{Gal}(C) = H$.

There exists some $s_j | r_j$ (where, potentially, some of the $s_j = 1$) such that

$$H \cong \mathbb{Z}/s_1\mathbb{Z} \times \cdots \times \mathbb{Z}/s_n\mathbb{Z}.$$

Let $\vec{\alpha}_j \in H$ be a generating set of H such that the order of α_j is s_j . Therefore, if $\vec{\alpha} \in \mathcal{R}^*$, we can find α_j^* such that $0 \leq \alpha_j^* \leq s_j - 1$ and

$$\vec{\alpha} = \sum_{j=1}^n \alpha_j^* \vec{\alpha}_j.$$

If we let $\vec{\alpha}_j = (\alpha_{j,1}, \dots, \alpha_{j,n})$ then for all $\vec{\alpha} \in \mathcal{R}^*$,

$$\alpha_k = \sum_{j=1}^n \alpha_j^* \alpha_{j,k}.$$

Now, a basis element of $K(C) = K\left(\sqrt[r_1]{F_1(X)}, \dots, \sqrt[r_n]{F_n(X)}\right)$ will be

$$\prod_{k=1}^n F_k(X)^{\frac{m_k}{r_k}} = \prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\sum_{k=1}^n \frac{\alpha_k m_k}{r_k}} = \prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\sum_{k=1}^n \frac{m_k}{r_k} \sum_{j=1}^n \alpha_j^* \alpha_{j,k}}$$

for some values of $m_k, k = 1, \dots, n$. (Note, we can restrict the product down to the $\vec{\alpha} \in \mathcal{R}^*$ for if $\vec{\alpha} \notin \mathcal{R}^*$, then $\deg(f_{\vec{\alpha}}) = 0$ and hence $f_{\vec{\alpha}} = 1$.) Therefore, we can define an action by $h = (h_1, \dots, h_n) \in H$ on the basis elements by

$$h \left(\prod_{k=1}^n F_k(X)^{\frac{m_k}{r_k}} \right) = \prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\sum_{k=1}^n \frac{m_k}{r_k} \sum_{j=1}^n h_j \alpha_j^* \alpha_{j,k}}.$$

Therefore if $h \neq (0, \dots, 0)$, there will be a $\vec{\alpha} \in \mathcal{R}^*$ such that

$$\sum_{k=1}^n \frac{m_k}{r_k} \sum_{j=1}^n h_j \alpha_j^* \alpha_{j,k} \notin \mathbb{Z}.$$

Hence, every non-trivial element of H gives a non-trivial automorphism of $K(C)$ and $H \subset \text{Gal}(K(C)/K) = \text{Gal}(C)$.

Define

$$F_j^* = \prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\alpha_j^*}, j = 1, \dots, n.$$

Since $\vec{\alpha}_j$ has order s_j we get $s_j(\vec{\alpha}_j) = (0, \dots, 0)$. Therefore, $s_j \alpha_{j,k} \equiv 0 \pmod{r_k}$ and we can find $\alpha'_{j,k}$ such that

$$\alpha_{j,k} = \frac{r_k}{(s_j, r_k)} \alpha'_{j,k}.$$

Therefore,

$$\begin{aligned} \sqrt[r_k]{F_k(X)} &= \prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\alpha_k/r_k} = \prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\frac{1}{r_k} \sum_{j=1}^n \alpha_j^* \alpha_{j,k}} = \prod_{j=1}^n \left(\prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\alpha_j^*} \right)^{\alpha_{j,k}/r_k} \\ &= \prod_{j=1}^n \left(\prod_{\vec{\alpha} \in \mathcal{R}^*} f_{\vec{\alpha}}^{\alpha_j^*} \right)^{\alpha'_{j,k}/(s_j, r_k)} = \prod_{j=1}^n \left(\sqrt[r_k]{F_j^*(X)} \right)^{\alpha'_{j,k} s_j / (s_j, r_k)}. \end{aligned}$$

Hence,

$$K \left(\sqrt[r_1]{F_1(X)}, \dots, \sqrt[r_n]{F_n(X)} \right) \subset K \left(\sqrt[r_1]{F_1^*(X)}, \dots, \sqrt[r_n]{F_n^*(X)} \right).$$

Clearly $\text{Gal} \left(K \left(\sqrt[r_1]{F_1^*(X)}, \dots, \sqrt[r_n]{F_n^*(X)} \right) / K \right) \subset H$. Thus $\text{Gal}(C) \subset H$ and therefore $\text{Gal}(C) = H$.

It remains to show that $g(C) = \frac{g-1}{|G|/|H|} + 1$.

Recall, $e(\vec{\alpha}) = \text{lcm} \left(\frac{r_j}{(r_j, \alpha_j)} \right)$. Then $e(\vec{\alpha})$ will be the order of $\vec{\alpha}$ as viewed as an element in G . Therefore, if $\vec{\alpha} \in \mathcal{R}^*$, then

$$e(\vec{\alpha}) = \text{lcm} \left(\frac{r_j}{(r_j, \alpha_j)} \right) = \text{lcm} \left(\frac{s_i}{(s_i, \alpha_i^*)} \right) := e^*(\vec{\alpha})$$

since $e^*(\vec{\alpha})$ would be the order of $\vec{\alpha}^*$ as viewed as an element in H (which would be the same as $\vec{\alpha}$ in G). Therefore,

$$c(\vec{\alpha}) = |G| - \frac{|G|}{e(\vec{\alpha})} = \frac{|G|}{|H|} \left(|H| - \frac{|H|}{e^*(\vec{\alpha})} \right) := \frac{|G|}{|H|} c^*(\vec{\alpha}).$$

Likewise, if we define $d_j^* = \deg(F_j^*) = \sum_{\vec{\alpha} \in \mathcal{R}^*} \alpha_j^* d(\vec{\alpha})$, then $e^*(\vec{d}^*) = e(\vec{d})$ and $\frac{|G|}{|H|} c^*(\vec{d}^*) = c(\vec{d})$.

Since $\vec{d}(\vec{\alpha})$ satisfies (2.1), we have

$$\begin{aligned} 2g + 2|G| - 2 &= \sum_{\vec{\alpha} \in \mathcal{R}} c(\vec{\alpha})d(\vec{\alpha}) + c(\vec{d}) = \sum_{\vec{\alpha} \in \mathcal{R}^*} c(\vec{\alpha})d(\vec{\alpha}) + c(\vec{d}) \\ &= \frac{|G|}{|H|} \left(\sum_{\vec{\alpha} \in \mathcal{R}^*} c^*(\vec{\alpha})d(\vec{\alpha}) + c^*(\vec{d}^*) \right). \end{aligned}$$

That is,

$$\left(\sum_{\vec{\alpha} \in \mathcal{R}^*} c^*(\vec{\alpha})d(\vec{\alpha}) + c^*(\vec{d}^*) \right) = 2 \left(\frac{g-1}{|G|/|H|} + 1 \right) + 2|H| - 2$$

Therefore, $(f_{\vec{\alpha}})$ corresponds to a monic curve C with $\text{Gal}(C) = H$ and, by the Riemann-Hurwitz formula, $g(C)$ is $\frac{g-1}{|G|/|H|} + 1$. \square

Therefore, for any monic curve with $\text{Gal}(C) = H \subset G$ and $g(C) = \frac{g-1}{|G|/|H|} + 1$, we can find $(f_{\vec{\alpha}}) \in \mathcal{F}_{\vec{d}(\vec{\alpha})}$ such that $K(C) = K(\sqrt[r_1]{F_1(X)}, \dots, \sqrt[r_\ell]{F_\ell(X)})$ where

$$F_j(X) = \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}}(X)^{\alpha_j}.$$

Corollary 3.3. *For any $\vec{k} \in \mathcal{R}'$ and E as in (2.6), let*

$$\begin{aligned} M_{\vec{k}, E}(G, g) &= \{C, \text{ monic} : \text{Gal}(C) = H \subset G, g(C) = \frac{g-1}{|G|/|H|} + 1, \deg F_j \equiv k_j \pmod{r_j} \\ &\quad \chi_{r_j}(F_j(x_i)) = \epsilon_{i,j}, i = 1, \dots, \ell, j = 1, \dots, n\}. \end{aligned}$$

Then there is a natural bijection from elements of $\mathcal{F}_{D; \vec{k}, E}$ to $M_{G, g}(\vec{k}, E)$ where $D = 2g + 2|G| - 2 - c(\vec{k})$.

Proof. Follows immediately from Proposition 3.2 and the definition of $\mathcal{F}_{D; \vec{k}, E}$. \square

4. INCLUSION-EXCLUSION OF ABELIAN GROUPS

Therefore, if we can determine the size of $\mathcal{F}_{D; \vec{k}, E}$ corresponding to curves with any abelian Galois group and any genus, then we can hope to do an inclusion-exclusion type argument for abelian groups. Luckily, this was first done by Delsarte [4].

Let \mathcal{G} be the set of all abelian groups. Define a function

$$\mu : \mathcal{G} \rightarrow \mathbb{Z}$$

by

$$\mu(\mathbb{Z}/p^{a_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{a_n}\mathbb{Z}) = \begin{cases} (-1)^n p^{\frac{n(n-1)}{2}} & a_1 = \dots = a_n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

To finish the definition if $G = G_1 \times G_2$ such that $(|G_1|, |G_2|) = 1$, then $\mu(G) = \mu(G_1)\mu(G_2)$. Then we have the property that

$$(4.1) \quad \sum_{H \subset G} \mu(H) = \begin{cases} 1 & G = \{e\} \\ 0 & \text{otherwise} \end{cases}.$$

Remark 4.1. This formula requires that we sum up over all subgroups of G in the sense of Remark 3.1. Hence why it is important that we define $M_{G,g}$ and $M_{G,g}(\vec{k}, E)$ in the way that we do.

For an example of (4.1) consider the group $\mathbb{Z}/Q^2\mathbb{Z}$, for Q a prime. Then the subgroups are $\{e\}$, $\mathbb{Z}/Q\mathbb{Z}$ and $\mathbb{Z}/Q^2\mathbb{Z}$ and each of them appear once. Therefore,

$$\begin{aligned} \sum_{H \subset \mathbb{Z}/Q^2\mathbb{Z}} \mu(H) &= \mu(\{e\}) + \mu(\mathbb{Z}/Q\mathbb{Z}) + \mu(\mathbb{Z}/Q^2\mathbb{Z}) \\ &= 1 + (-1) + 0 = 0. \end{aligned}$$

Whereas if we consider the group $\mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$, for Q a prime, then the subgroups would be $\{e\}$, $\mathbb{Z}/Q\mathbb{Z}$ and $\mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$. Obviously $\{e\}$ and $\mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$ appear only once however, $\mathbb{Z}/Q\mathbb{Z}$ can appear many times. It is easy to see that all the subgroups of $\mathbb{Z}/Q\mathbb{Z}$ lying in $\mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$ will be generated by $(1, a)$, $a \in \mathbb{Z}/Q\mathbb{Z}$ or $(0, 1)$. That is, there are $Q+1$ different subgroups of $\mathbb{Z}/Q\mathbb{Z}$ appearing in $\mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$. Therefore,

$$\begin{aligned} \sum_{H \subset \mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}} \mu(H) &= \mu(\{e\}) + (Q+1)\mu(\mathbb{Z}/Q\mathbb{Z}) + \mu(\mathbb{Z}/Q\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}) \\ &= 1 + (Q+1)(-1) + Q = 0. \end{aligned}$$

This allows us to perform Möbius inversion on $M(G, g)$.

Lemma 4.2. *For any abelian group, G , and genus, g ,*

$$|\mathcal{H}_{G,g}^*| = \sum_{H \subset G} \mu(G/H) |M(H, \frac{g-1}{|G|/|H|} + 1)|$$

Likewise, for any $\vec{k} \in \mathcal{R}$ and E as in (2.6)

$$|\mathcal{H}_{G,g}^*(\vec{k}, E)| = \sum_{H \subset G} |M_{\vec{k},E}(H, \frac{g-1}{|G|/|H|} + 1)|$$

Proof. Straight from the definition we get

$$|M(G, g)| = \sum_{H \subset G} \left| \mathcal{H}_{H, \frac{g-1}{|G|/|H|} + 1}^* \right|.$$

Therefore,

$$\begin{aligned} \sum_{H \subset G} \mu(G/H) \left| M\left(H, \frac{g-1}{|G|/|H|} + 1\right) \right| &= \sum_{H \subset G} \mu(G/H) \sum_{H' \subset H} \left| \mathcal{H}_{H', \frac{g-1}{|G|/|H'|} + 1}^* \right| \\ &= \sum_{H' \subset G} \left| \mathcal{H}_{H', \frac{g-1}{|G|/|H'|} + 1}^* \right| \sum_{H' \subset H \subset G} \mu(G/H) \\ &= \sum_{H' \subset G} \left| \mathcal{H}_{H', \frac{g-1}{|G|/|H'|} + 1}^* \right| \sum_{H'' \subset G/H'} \mu(H'') \\ &= |\mathcal{H}_{G,g}^*|. \end{aligned}$$

The proof of the likewise is analogous. □

5. GENERATING SERIES

It remains to determine the size of $\mathcal{F}_{D;\vec{k},E}$ as $D \rightarrow \infty$. In order to do this we will develop a generating series for this set. But first, we need indicator functions for the relations

$$d_j \equiv k_j \pmod{r_j}, j = 1, \dots, n$$

$$\chi_{r_j}(F(x_i)) = \epsilon_{i,j}, i = 1, \dots, \ell, j = 1, \dots, n.$$

That is, if we let $\xi_{r_j} = e^{\frac{2\pi i}{r_j}}$, a primitive r_j^{th} root of unity, then

(5.1)

$$\frac{1}{r_1 \cdots r_n} \prod_{j=1}^n \sum_{t_j=0}^{r_j-1} \xi_{r_j}^{t_j(\sum \alpha_j \deg(f_{\vec{\alpha}}) - k_j)} = \begin{cases} 1 & \sum_{\vec{\alpha} \in \mathcal{R}} \alpha_j \deg(f_{\vec{\alpha}}) \equiv k_j \pmod{r_j} \\ 0 & \text{otherwise} \end{cases}.$$

Further, if we denote $h(X) = \prod_{i=1}^{\ell} (X - x_i)$, then as long as $(F_j, h) = 1$ for $j = 1, \dots, n$, we get

(5.2)

$$\left(\frac{1}{r_1 \cdots r_n} \right)^{\ell} \prod_{i=1}^{\ell} \prod_{j=1}^n \sum_{\nu_{i,j}=0}^{r_j-1} (\epsilon_{i,j}^{-1} \chi_{r_j}(F_j(x_i)))^{\nu_{i,j}} = \begin{cases} 1 & \chi_{r_j}(F_j(x_i)) = \epsilon_{i,j}, i = 1, \dots, \ell, j = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}.$$

Remark 5.1. The sum in the exponent in (5.1) is a sum over all $\vec{\alpha} \in \mathcal{R}$.

For ease of notation, for every set of polynomials $(f_{\vec{\alpha}})$, let $I_{\vec{k},E}((f_{\vec{\alpha}}))$ be the indicator function defined as

(5.3)

$$I_{\vec{k},E}((f_{\vec{\alpha}})) = \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell+1} \left(\prod_{j=1}^n \sum_{t_j=0}^{r_j-1} \xi_{r_j}^{t_j(\sum \alpha_j \deg(f_{\vec{\alpha}}) - k_j)} \right) \left(\prod_{i=1}^{\ell} \prod_{j=1}^n \sum_{\nu_{i,j}=0}^{r_j-1} (\epsilon_{i,j}^{-1} \chi_{r_j}(F_j(x_i)))^{\nu_{i,j}} \right).$$

Now, define the multi-variable complex function

$$(5.4) \quad \mathcal{G}_{\vec{k},E}((s_{\vec{\alpha}})) = \sum_{(f_{\vec{\alpha}})} \frac{\mu^2(h \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}}) I_{\vec{k},E}((f_{\vec{\alpha}}))}{\prod_{\vec{\alpha} \in \mathcal{R}} |f_{\vec{\alpha}}|^{c(\vec{\alpha}) s_{\vec{\alpha}}}}.$$

Remark 5.2. The sum is over all $r_1 \cdots r_n - 1$ -tuples of monic polynomials $(f_{\vec{\alpha}})_{\vec{\alpha} \in \mathcal{R}}$. However, the factor $\mu^2(h \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}})$ means that it is zero whenever the set of polynomials $(f_{\vec{\alpha}})$ are not square-free and pairwise coprime as well as coprime to h (and thus non-zero at any of the x_i). Moreover, as usual, we let $|f_{\vec{\alpha}}| = q^{\deg(f_{\vec{\alpha}})}$.

Now, if we let $z_{\vec{\alpha}} = q^{-s_{\vec{\alpha}}}$ and define $F_{\vec{k},E}((z_{\vec{\alpha}})) = \mathcal{G}_{\vec{k},E}((q^{-s_{\vec{\alpha}}}))$, then

$$\begin{aligned} F_{\vec{k},E}((z_{\vec{\alpha}})) &= \sum_{(f_{\vec{\alpha}})} \mu^2(h \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}}) I_{\vec{k},E}((f_{\vec{\alpha}})) \prod_{\vec{\alpha} \in \mathcal{R}} z_{\vec{\alpha}}^{c(\vec{\alpha}) \deg(f_{\vec{\alpha}})} \\ &= \sum_{\substack{d(\vec{\alpha})=0 \\ \vec{\alpha} \in \mathcal{R}}}^{\infty} |\mathcal{F}_{\vec{d}(\vec{\alpha}); \vec{k}, E}| \prod_{\vec{\alpha} \in \mathcal{R}} z_{\vec{\alpha}}^{c(\vec{\alpha}) d(\vec{\alpha})}. \end{aligned}$$

With some abuse of notation, if we let $F_{\vec{k},E}(z)$ be the function that sets all the $z_{\vec{\alpha}} = z$ to be the same in $F_{\vec{k},E}((z_{\vec{\alpha}}))$, then we get

$$(5.5) \quad \begin{aligned} F_{\vec{k},E}(z) &= \sum_{\substack{d(\vec{\alpha})=0 \\ \vec{\alpha} \in \mathcal{R}}}^{\infty} |\mathcal{F}_{\vec{d}(\vec{\alpha});\vec{k},E}| z^{\sum_{\vec{\alpha} \in \mathcal{R}} c(\vec{\alpha})d(\vec{\alpha})} \\ &= \sum_{D=0}^{\infty} |\mathcal{F}_{D;\vec{k},E}| z^D. \end{aligned}$$

Ideally, we would like to write $F_{\vec{k},E}(z)$ as an Euler product. However, this is not possible. We can, though, write it as a sum of functions that can be written as a Euler product. But first we need some notation. Let

$$\mathcal{M} := \left\{ \nu = \begin{pmatrix} \nu_{1,1} & \cdots & \nu_{1,n} \\ \vdots & & \vdots \\ \nu_{\ell,1} & \cdots & \nu_{\ell,n} \end{pmatrix} \in M_{\ell,n} : \nu_{i,j} \in \mathbb{Z}/r_j\mathbb{Z} \right\}.$$

We can define an action on \mathcal{R}' and E by \mathcal{M} by

$$(5.6) \quad \nu \vec{\alpha} := \begin{pmatrix} \sum_{j=1}^n \frac{r_n}{r_j} \nu_{1,j} \alpha_j \\ \vdots \\ \sum_{j=1}^n \frac{r_n}{r_j} \nu_{\ell,j} \alpha_j \end{pmatrix} \in (\mathbb{Z}/r_n\mathbb{Z})^{\ell}$$

$$(5.7) \quad E^{\nu} := \prod_{i=1}^{\ell} \prod_{j=1}^n \epsilon_{i,j}^{\nu_{i,j}} \in \mu_{r_n}$$

Moreover, for any $\vec{\alpha}, \vec{\beta} \in \mathcal{R}'$ define

$$(5.8) \quad \vec{\alpha} \cdot \vec{\beta} = \sum_{j=1}^n \frac{r_n}{r_j} \alpha_j \beta_j \in \mathbb{Z}/r_n\mathbb{Z}.$$

With this notation, we can rewrite (5.1) as

$$\begin{aligned} \frac{1}{r_1 \cdots r_n} \prod_{j=1}^n \sum_{t_j=0}^{r_j-1} \xi_{r_j}^{t_j(\sum \alpha_j \deg(f_{\vec{\alpha}}) - k_j)} &= \frac{1}{r_1 \cdots r_n} \sum_{\vec{t} \in \mathcal{R}'} \prod_{j=1}^n \xi_{r_j}^{t_j(\sum \alpha_j \deg(f_{\vec{\alpha}}) - k_j)} \\ &= \frac{1}{r_1 \cdots r_n} \sum_{\vec{t} \in \mathcal{R}'} \xi_{r_n}^{-\vec{t} \cdot \vec{k}} \prod_{\vec{\alpha} \in \mathcal{R}} \xi_{r_n}^{\vec{t} \cdot \vec{\alpha} \deg(f_{\vec{\alpha}})}. \end{aligned}$$

Recall $h(X) = \prod_{i=1}^{\ell} (X - x_i)$. For every $\nu \in \mathcal{M}$ and $\vec{\alpha} \in \mathcal{R}$, define

$$(5.9) \quad \chi_{r_n}^{\nu \vec{\alpha}}(F(X)) = \begin{cases} \prod_{i=1}^{\ell} \chi_{r_n}^{(\nu \vec{\alpha})_i}(F(x_i)) & (F, h) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then, $\chi_{r_n}^{\nu \vec{\alpha}}$ is a multiplicative character on $\mathbb{F}_q[X]$ modulo $h(X)$. Moreover, it will be trivial if and only if $\nu \vec{\alpha} = \vec{0}$. Hence, we can rewrite (5.2) as

$$\left(\frac{1}{r_1 \cdots r_n} \right)^{\ell} \prod_{i=1}^{\ell} \prod_{j=1}^n \sum_{\nu_{i,j}=0}^{r_j-1} (\epsilon_{i,j}^{-1} \chi_{r_j}(F_j(x_i)))^{\nu_{i,j}} = \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell} \sum_{\nu \in \mathcal{M}} \prod_{i=1}^{\ell} \prod_{j=1}^n (\epsilon_{i,j}^{-1} \chi_{r_j}(F_j(x_i)))^{\nu_{i,j}}$$

$$= \left(\frac{1}{r_1 \cdots r_n} \right)^\ell \sum_{\nu \in \mathcal{M}} E^{-\nu} \prod_{\vec{\alpha} \in \mathcal{R}} \prod_{i=1}^\ell \prod_{j=1}^n \chi_{r_j}^{\nu_{i,j}}(f_{\vec{\alpha}}^{\alpha_j}(x_i)) = \left(\frac{1}{r_1 \cdots r_n} \right)^\ell \sum_{\nu \in \mathcal{M}} E^{-\nu} \prod_{\vec{\alpha} \in \mathcal{R}} \chi_{r_n}^{\nu \vec{\alpha}}(f_{\vec{\alpha}}(X)).$$

Therefore, we can rewrite the indicator function in (5.3) as

$$(5.10) \quad I_{\vec{k}, E}((f_{\vec{\alpha}})) = \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell+1} \sum_{\vec{t} \in \mathcal{R}'} \sum_{\nu \in \mathcal{M}} E^{-\nu} \xi_{r_n}^{-\vec{t} \cdot \vec{k}} \prod_{\vec{\alpha} \in \mathcal{R}} \xi_{r_n}^{\vec{t} \cdot \vec{\alpha} \deg(f_{\vec{\alpha}})} \chi_{r_n}^{\nu \vec{\alpha}}(f_{\vec{\alpha}}(X)).$$

We can rewrite $F_{\vec{k}, E}(z)$ using this new notation.

$$\begin{aligned} F_{\vec{k}, E}(z) &= \sum_{(f_{\vec{\alpha}})} \mu^2(h \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}}) I_{\vec{k}, E}((f_{\vec{\alpha}})) z^{\sum c(\vec{\alpha}) \deg(f_{\vec{\alpha}})} \\ &= \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell+1} \sum_{(f_{\vec{\alpha}})} \mu^2(h \prod_{\vec{\alpha} \in \mathcal{R}} f_{\vec{\alpha}}) \sum_{\vec{t} \in \mathcal{R}'} \sum_{\nu \in \mathcal{M}} E^{-\nu} \xi_{r_n}^{-\vec{t} \cdot \vec{k}} \prod_{\vec{\alpha} \in \mathcal{R}} \left(\chi_{r_n}^{\nu \vec{\alpha}}(f_{\vec{\alpha}}) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(f_{\vec{\alpha}})} \right) \\ &= \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell+1} \sum_{\vec{t} \in \mathcal{R}'} \sum_{\nu \in \mathcal{M}} E^{-\nu} \xi_{r_n}^{-\vec{t} \cdot \vec{k}} A_{\vec{t}, \nu}(z) \end{aligned}$$

where

$$A_{\vec{t}, \nu}(z) := \sum_{(f_{\vec{\alpha}})} \mu^2(h \prod_{\vec{\alpha}} f_{\vec{\alpha}}) \prod_{\vec{\alpha} \in \mathcal{R}} \left(\chi_{r_n}^{\nu \vec{\alpha}}(f_{\vec{\alpha}}) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(f_{\vec{\alpha}})} \right).$$

Definition 5.3. We call a function $G : \mathbb{F}_q[X]^n \rightarrow \mathbb{C}$ an **n -dimensional multiplicative** function if

$$G(f_1, \dots, f_n) = \prod_P G(P^{v_P(f_1)}, \dots, P^{v_P(f_n)})$$

where the product is over all prime polynomial P dividing $f_1 \cdots f_n$.

Therefore, if G is an n -dimensional multiplicative function, then

$$\sum_{f_1, \dots, f_n} G(f_1, \dots, f_n) = \prod_P \left(1 + \sum_{(a_1, \dots, a_n) \neq (0, \dots, 0)} G(P^{a_1}, \dots, P^{a_n}) \right).$$

where the sum is over all monic polynomials in $\mathbb{F}_q[X]$ and the product is over all monic prime polynomials.

Now,

$$G((f_{\vec{\alpha}})) = \mu^2(h \prod_{\vec{\alpha}} f_{\vec{\alpha}}) \prod_{\vec{\alpha} \in \mathcal{R}} \left(\chi_{r_n}^{\nu \vec{\alpha}}(f_{\vec{\alpha}}) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(f_{\vec{\alpha}})} \right)$$

is an $|\mathcal{R}|$ -dimensional multiplicative function. Moreover, if P is a prime polynomial coprime to h , then

$$G((P^{a_{\vec{\alpha}}})) = \begin{cases} \chi_{r_n}^{\nu \vec{\alpha}_0}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}_0} z^{c(\vec{\alpha}_0)})^{\deg(P)} & a_{\vec{\alpha}_0} = 1 \text{ for some } \vec{\alpha}_0, a_{\vec{\beta}} = 0 \text{ for all } \vec{\beta} \neq \vec{\alpha}_0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} A_{\vec{t},\nu}(z) &= \sum_{\substack{f_{\vec{\alpha}} \\ \vec{\alpha} \in \mathcal{R}}} \mu^2(h) \prod_{\vec{\alpha}} f_{\vec{\alpha}} \prod_{\vec{\alpha} \in \mathcal{R}} \left(\chi_{r_n}^{\nu\vec{\alpha}}(f_{\vec{\alpha}}) (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(f_{\vec{\alpha}})} \right) \\ &= \prod_{\substack{P \\ (P,h)=1}} \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}} \chi_{r_n}^{\nu\vec{\alpha}}(P) (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right). \end{aligned}$$

Now, if we let $c_1 < c_2 < \dots < c_\eta$ be the unique values of the $c(\vec{\alpha})$, then we see that if $|z| < q^{-1/c_1}$, then $A_{\vec{t},\nu}(z)$ absolutely converges for all \vec{t}, ν and, hence, so does $F_{\vec{k},E}(z)$. Therefore, we can express $|\mathcal{F}_{D;\vec{k},E}|$ as a contour integral of $F_{\vec{k},E}(z)$.

Proposition 5.4. *If $c_1 = \min(c(\vec{\alpha}))$ and $0 < \delta_1 < q^{-1/c_1}$ then let $C_{\delta_1} = \{z \in \mathbb{C} : |z| = \delta_1\}$, oriented counterclockwise. Then*

$$(5.11) \quad \frac{1}{2\pi i} \oint_{C_{\delta_1}} \frac{F_{\vec{k},E}(z)}{z^{D+1}} dz = |\mathcal{F}_{D;\vec{k},E}|.$$

Proof. By (5.5), we have

$$F_{\vec{k},E}(z) = \sum_{D=0}^{\infty} |\mathcal{F}_{D;\vec{k},E}| z^D.$$

By our discussion above, $\frac{F_{\vec{k},E}(z)}{z^{D+1}}$ has only one pole at 0 in the region contained in C_{δ_1} and it's residue is $|\mathcal{F}_{D;\vec{k},E}|$. □

6. ANALYTIC CONTINUATION OF $A_{\vec{t},\nu}(z)$

In this section, we will calculate an analytic continuation for $A_{\vec{t},\nu}(z)$ for all \vec{t}, ν . Then in the next section, we will use this analytic continuation to analyze the poles of $A_{\vec{t},\nu}(z)$.

Recall $h(X) = \prod_{i=1}^{\ell} (X - x_i)$ and define $\mathcal{R}_\nu = \{\vec{\alpha} \in \mathcal{R} : \nu\vec{\alpha} = \vec{0}\}$, then the character $\chi_{r_n}^{\nu\vec{\alpha}}$ (as defined in (5.9)) will be trivial if and only if $\vec{\alpha} \in \mathcal{R}_\nu$. Therefore,

$$\begin{aligned} A_{\vec{t},\nu}(z) &= \prod_{\substack{P \\ (P,h)=1}} \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}} \chi_{r_n}^{\nu\vec{\alpha}}(P) (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) \\ &= \prod_{\substack{P \\ (P,h)=1}} \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} + \sum_{\vec{\alpha} \notin \mathcal{R}_\nu} \chi_{r_n}^{\nu\vec{\alpha}}(P) (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) \\ &= \prod_P \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} + \sum_{\vec{\alpha} \notin \mathcal{R}_\nu} \chi_{r_n}^{\nu\vec{\alpha}}(P) (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) \times \\ &\quad \prod_{P|h} \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right)^{-1} \\ &= \prod_{\vec{\alpha} \in \mathcal{R}_\nu} \prod_P \left(1 + (\xi_{r_n}^{\vec{t}\cdot\vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) H_{\vec{t},\nu}(z) \end{aligned}$$

$$= \prod_{\vec{\alpha} \in \mathcal{R}_\nu} \frac{Z_K(\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})}{Z_K(\xi_{r_n}^{2\vec{t} \cdot \vec{\alpha}} z^{2c(\vec{\alpha})})} H_{\vec{t}, \nu}(z)$$

where

$$Z_K(z) = \prod_P \left(1 - z^{\deg(P)}\right)^{-1} = (1 - qz)^{-1}$$

is the zeta-function of K in the z -variable and

$$H_{\vec{t}, \nu}(z) = \prod_P \left(\frac{1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} + \sum_{\vec{\alpha} \notin \mathcal{R}_\nu} \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}}{\prod_{\vec{\alpha} \in \mathcal{R}_\nu} \left(1 + (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}\right)} \right) \times \\ \prod_{P|h} \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right)^{-1}.$$

Now, for all $\vec{\alpha} \in \mathcal{R}$,

$$\frac{Z_K(\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})}{Z_K(\xi_{r_n}^{2\vec{t} \cdot \vec{\alpha}} z^{2c(\vec{\alpha})})}$$

is a meromorphic function with simple poles when $z^{c(\vec{\alpha})} = (q \xi_{r_n}^{\vec{t} \cdot \vec{\alpha}})^{-1/c(\vec{\alpha})}$. So it remains to determine where $H_{\vec{t}, \nu}(z)$ converges.

Lemma 6.1. $H_{\vec{t}, \nu}(z)$ absolutely converges for all $|z| < q^{-1/2c_1}$.

Proof. Since

$$\prod_{P|h} \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right)^{-1}$$

is a finite product, it will always converge and thus we need only consider the factor consisting of the infinite product.

$$\prod_P \left(\frac{1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} + \sum_{\vec{\alpha} \notin \mathcal{R}_\nu} \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}}{\prod_{\vec{\alpha} \in \mathcal{R}_\nu} \left(1 + (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}\right)} \right) \\ = \prod_{\vec{\alpha} \notin \mathcal{R}_\nu} \prod_P \left(1 + \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) H_{\vec{t}, \nu}^*(z).$$

Since, for all $\vec{\alpha} \notin \mathcal{R}_\nu$, $\chi_{r_n}^{\nu \vec{\alpha}}$ is a non-trivial character we get that

$$\prod_{\vec{\alpha} \notin \mathcal{R}_\nu} \prod_P \left(1 + \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right)$$

is an entire function. Moreover,

$$H_{\vec{t}, \nu}^*(z) = \prod_P \left(\frac{1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} + \sum_{\vec{\alpha} \notin \mathcal{R}_\nu} \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}}{\prod_{\vec{\alpha} \in \mathcal{R}_\nu} \left(1 + (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}\right) \prod_{\vec{\alpha} \notin \mathcal{R}_\nu} \left(1 + \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}\right)} \right) \\ = \prod_P \left(1 - \frac{h_P(z)}{\prod_{\vec{\alpha} \in \mathcal{R}_\nu} \left(1 + (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}\right) \prod_{\vec{\alpha} \notin \mathcal{R}_\nu} \left(1 + \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)}\right)} \right)$$

where

$$\begin{aligned} h_p(z) &= \prod_{\vec{\alpha} \in \mathcal{R}_\nu} \left(1 + (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) \prod_{\vec{\alpha} \notin \mathcal{R}_\nu} \left(1 + \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) \\ &\quad - \left(1 + \sum_{\vec{\alpha} \in \mathcal{R}_\nu} (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} + \sum_{\vec{\alpha} \notin \mathcal{R}_\nu} \chi_{r_n}^{\nu \vec{\alpha}}(P) (\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})^{\deg(P)} \right) \\ &= O \left(z^{\frac{\min(c(\vec{\alpha}) + c(\vec{\beta}))}{\vec{\alpha} \neq \vec{\beta}}} \right) = O(z^{2c_1}). \end{aligned}$$

Therefore, if $|z| < q^{-1/2c_1}$, then $H_{\vec{t}, \nu}^*(z)$ converges absolutely and hence so does $H_{\vec{t}, \nu}(z)$. \square

For $0 \leq a \leq r_n - 1$, and $i = 1, \dots, \eta$, define

$$(6.1) \quad \mathcal{R}_{\vec{t}, \nu; a, i} = \{\vec{\alpha} \in \mathcal{R}_\nu : c(\vec{\alpha}) = c_i \text{ and } \vec{t} \cdot \vec{\alpha} \equiv a \pmod{r_n}\}$$

and let

$$(6.2) \quad m_{\vec{t}, \nu; a, i} = |\mathcal{R}_{\vec{t}, \nu; a, i}|.$$

Corollary 6.2. $A_{\vec{t}, \nu}(z)$ is meromorphic on the disc $|z| < q^{-1/2c_1}$ with poles of order $m_{\vec{t}, \nu; a, i}$ at

$$z = \xi_{c_i}^k (q \xi_{r_n}^a)^{-1/c_i}$$

for $k = 1, \dots, c_i$.

Proof. Immediate from Lemma 6.1 and the factors of $Z_K(z)$ appearing. \square

Remark 6.3. It is highly possible that $m_{\vec{t}, \nu; a, i} = 0$ for some values of \vec{t}, ν, a, i . In this case when we say a pole of order 0, we mean there is no pole.

7. RESIDUE CALCULATIONS

Now, we can calculate the residues of $A_{\vec{t}, \nu}(z)$ at each of its poles.

Lemma 7.1. Let a, i be such that $m_{\vec{t}, \nu; a, i} \neq 0$, then for any $1 \leq k \leq c_i$,

$$\text{Res}_{z = \xi_{c_i}^k (q \xi_{r_n}^a)^{-1/c_i}} \left(\frac{A_{\vec{t}, \nu}(z)}{z^{D+1}} \right) = P_{\vec{t}, \nu; a, i, k}(D) q^{\frac{D}{c_i}}$$

where $P_{\vec{t}, \nu; a, i, k}$ is a quasi-polynomial of degree $(m_{\vec{t}, \nu; a, i} - 1)$ with leading coefficient $-C_{\vec{t}, \nu; a, i, k}$ such that

$$C_{\vec{t}, \nu; a, i, k} = \frac{1}{(m_{\vec{t}, \nu; a, i} - 1)!} \left(\frac{1 - q^{-1}}{c_i} \right)^{m_{\vec{t}, \nu; a, i}} \xi_{c_i}^{-kD} (\xi_{r_n}^a)^{\frac{D}{c_i}} H_{\vec{t}, \nu; a, i}(\xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i})$$

and $H_{\vec{t}, \nu; a, i}$ is defined in the proof.

Proof.

$$\frac{A_{\vec{t}, \nu}(z)}{z^{D+1}} = \frac{1}{z^{D+1}} \prod_{\vec{\alpha} \in \mathcal{R}_\nu} \frac{Z_K(\xi_{r_n}^{\vec{t} \cdot \vec{\alpha}} z^{c(\vec{\alpha})})}{Z_K(\xi_{r_n}^{2\vec{t} \cdot \vec{\alpha}} z^{2c(\vec{\alpha})})} H_{\vec{t}, \nu}(z)$$

$$\begin{aligned}
 &= \frac{1}{z^{D+1}} \prod_{j=1}^{\eta} \prod_{\substack{b=0 \\ m_{\vec{t},\nu;b,j} \neq 0}}^{r_n-1} \left(\frac{Z_K(\xi_{r_n}^b z^{c_j})}{Z_K(\xi_{r_n}^{2b} z^{2c_j})} \right)^{m_{\vec{t},\nu;b,j}} H_{\vec{t},\nu}(z) \\
 &= \frac{1}{z^{D+1}} \left(\frac{1 - q \xi_{r_n}^{2a} z^{2c_i}}{1 - q \xi_{r_n}^a z^{c_i}} \right)^{m_{\vec{t},\nu;a,i}} H_{\vec{t},\nu;a,i}(z)
 \end{aligned}$$

where

$$H_{\vec{t},\nu;a,i}(z) = \prod_{\substack{j=1 \\ (b,j) \neq (a,i) \\ m_{\vec{t},\nu;b,j} \neq 0}}^{\eta} \prod_{b=0}^{r_n-1} \left(\frac{Z_K(\xi_{r_n}^b z^{c_j})}{Z_K(\xi_{r_n}^{2b} z^{2c_j})} \right)^{m_{\vec{t},\nu;b,j}} H_{\vec{t},\nu}(z).$$

Therefore, for any $1 \leq k \leq c_i$, if we let

$$R_{a,i,k}(z) = \frac{z^{c_i} - (q \xi_{r_n}^a)^{-1}}{z - \xi_{c_i}^k (q \xi_{r_n}^a)^{-1/c_i}},$$

then

$$\begin{aligned}
 & (m_{\vec{t},\nu;a,i} - 1)! \text{Res}_{z=\xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}} \left(\frac{A_{\vec{t},\nu}(z)}{z^{D+1}} \right) \\
 &= \lim_{z \rightarrow \xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}} \frac{d^{m_{\vec{t},\nu;a,i}-1}}{dz^{m_{\vec{t},\nu;a,i}-1}} \frac{(z - \xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i})^{m_{\vec{t},\nu;a,i}}}{z^{D+1}} \left(\frac{1 - q \xi_{r_n}^{2a} z^{2c_i}}{1 - q \xi_{r_n}^a z^{c_i}} \right)^{m_{\vec{t},\nu;a,i}} H_{\vec{t},\nu;a,i}(z) \\
 &= \lim_{z \rightarrow \xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}} \frac{d^{m_{\vec{t},\nu;a,i}-1}}{dz^{m_{\vec{t},\nu;a,i}-1}} \frac{1}{z^{D+1}} \left(\frac{1 - q \xi_{r_n}^{2a} z^{2c_i}}{-q \xi_{r_n}^a R_{a,i,k}(z)} \right)^{m_{\vec{t},\nu;a,i}} H_{\vec{t},\nu;a,i}(z) \\
 &= \lim_{z \rightarrow \xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}} \sum_{j=0}^{m_{\vec{t},\nu;a,i}-1} \binom{m_{\vec{t},\nu;a,i}-1}{j} \frac{d^j}{dz^j} \left(\frac{1}{z^{D+1}} \right) \frac{d^{m_{\vec{t},\nu;a,i}-1-j}}{dz^{m_{\vec{t},\nu;a,i}-1-j}} \left(\frac{1 - q \xi_{r_n}^{2a} z^{2c_i}}{-q \xi_{r_n}^a R_{a,i,k}(z)} \right)^{m_{\vec{t},\nu;a,i}} \times \\
 & \quad H_{\vec{t},\nu;a,i}(z) \\
 &= \sum_{j=0}^{m_{\vec{t},\nu;a,i}-1} \binom{m_{\vec{t},\nu;a,i}-1}{j} (-1)^j (D+1) \cdots (D+j) \xi_{c_i}^{-k(D+j+1)} (\xi_{r_n}^a q)^{(D+j+1)/c_i} \times \\
 & \quad \frac{d^{m_{\vec{t},\nu;a,i}-1-j}}{dz^{m_{\vec{t},\nu;a,i}-1-j}} \left(\frac{1 - q \xi_{r_n}^{2a} z^{2c_i}}{-q \xi_{r_n}^a R_{a,i,k}(z)} \right)^{m_{\vec{t},\nu;a,i}} H_{\vec{t},\nu;a,i}(z) \Big|_{z=\xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}} \\
 &= P_{\vec{t},\nu;a,i,k}(D) q^{\frac{D}{c_i}}
 \end{aligned}$$

where $P_{\vec{t},\nu;a,i,k}$ is a quasi-polynomial of degree $m_{\vec{t},\nu;a,i} - 1$. Moreover, we see that the leading coefficient of $P_{\vec{t},\nu;a,i,k}$ arises when $j = m_{\vec{t},\nu;a,i} - 1$. That is

$$\begin{aligned}
 P_{\vec{t},\nu;a,i,k}(D) q^{\frac{D}{c_i}} &= (-D)^{m_{\vec{t},\nu;a,i}-1} \xi_{c_i}^{-k(D+m_{\vec{t},\nu;a,i})} (\xi_{r_n}^a q)^{(D+m_{\vec{t},\nu;a,i})/c_i} \left(\frac{1 - q^{-1}}{-c_i \xi_{c_i}^{k(c_i-1)} (\xi_{r_n}^a q)^{1/c_i}} \right)^{m_{\vec{t},\nu;a,i}} \times \\
 & \quad H_{\vec{t},\nu;a,i}(\xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}) \left(1 + O\left(\frac{1}{D}\right) \right) \\
 &= - \left(\frac{1 - q^{-1}}{c_i} \right)^{m_{\vec{t},\nu;a,i}} \xi_{c_i}^{-kD} D^{m_{\vec{t},\nu;a,i}-1} (\xi_{r_n}^a q)^{\frac{D}{c_i}} H_{\vec{t},\nu;a,i}(\xi_{c_i}^k (\xi_{r_n}^a q)^{-1/c_i}) \times
 \end{aligned}$$

$$\left(1 + O\left(\frac{1}{D}\right)\right).$$

□

Corollary 7.2. Let $m_{\vec{t},\nu,i} = \max_{0 \leq a \leq r_n-1} (m_{\vec{t},\nu;a,i})$. Let $0 < \delta_1 < q^{-1/c_1}$, $\delta_2 = \frac{1+\epsilon}{2c_1}$ for some $\epsilon > 0$ and let $C_{\delta_1} = \{z \in \mathbb{C} : |z| = \delta_1\}$ oriented counterclockwise and $C_{\delta_2} = \{z \in \mathbb{C} : |z| = q^{-\delta_2}\}$ oriented clockwise. Then

$$\frac{1}{2\pi i} \oint_{C_{\delta_1} + C_{\delta_2}} \frac{A_{\vec{t},\nu}(z)}{z^{D+1}} dz = \sum_{i=1}^{\eta} P_{\vec{t},\nu,i}(D) q^{\frac{D}{c_i}}$$

where $P_{\vec{t},\nu,i}$ is a quasi-polynomial such that

$$P_{\vec{t},\nu,i}(D) = C_{\vec{t},\nu,i} D^{m_{\vec{t},\nu,i}-1} + O(D^{m_{\vec{t},\nu,i}-2})$$

with

$$C_{\vec{t},\nu,i} = \sum_{\substack{a=0 \\ m_{\vec{t},\nu;a,i}=m_{\vec{t},\nu,i}}}^{r_n-1} \sum_{k=0}^{c_i-1} C_{\vec{t},\nu;a,i,k}.$$

Proof. By Cauchy's Residue Theorem, and the fact that the larger disc, C_{δ_2} , is oriented clockwise,

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_{\delta_1} + C_{\delta_2}} \frac{A_{\vec{t},\nu}(z)}{z^{D+1}} dz &= \sum_{i=1}^{\eta} \sum_{\substack{a=0 \\ m_{\vec{t},\nu;a,i} \neq 0}}^{r_n-1} \sum_{k=0}^{c_i-1} -\text{Res}_{z=\xi_{c_i}^k (q\xi_{r_n}^a)^{-1/c_i}} \left(\frac{A_{\vec{t},\nu}(z)}{z^{D+1}} \right) \\ &= \sum_{i=1}^{\eta} \sum_{\substack{a=0 \\ m_{\vec{t},\nu;a,i} \neq 0}}^{r_n-1} \sum_{k=0}^{c_i-1} -P_{\vec{t},\nu;a,i,k}(D) q^{\frac{D}{c_i}} \\ &= \sum_{i=1}^{\eta} P_{\vec{t},\nu,i}(D) q^{\frac{D}{c_i}}. \end{aligned}$$

The fact that $P_{\vec{t},\nu,i}(D)$ satisfies the conditions in the statement follow directly from Lemma 7.1 □

Remark 7.3. Now, we are unable to determine if $C_{\vec{t},\nu,i}$ is non-zero. Hence we can only give a bound on the degree of the $P_{\vec{t},\nu,i}$. This is why in the statement of the main theorems we say "of degree at most" instead of give the exact degree.

Proposition 7.4. Let

$$m_i = \max_{\substack{\vec{t} \in \mathcal{R} \\ \nu \in \mathcal{M}}} (m_{\vec{t},\nu,i}).$$

If there exists a solution to (2.9) and (2.5), then for every $\epsilon > 0$,

$$|\mathcal{F}_{D,\vec{k},E}| = \sum_{i=1}^{\eta} P_i(D) q^{\frac{D}{c_i}} + O\left(q^{(\frac{1}{2}+\epsilon)\frac{D}{c_1}}\right)$$

where P_i is a quasi-polynomial such that of degree at most $(m_i - 1)$. Otherwise, if there does not exist a solution to (2.9) and (2.5), then $|\mathcal{F}_{D,\vec{k},E}| = 0$.

Proof. Recall that

$$\mathcal{F}_{D;\vec{k},E} = \bigcup_{\vec{d}(\vec{\alpha})} \mathcal{F}_{\vec{d}(\vec{\alpha});\vec{k},E}$$

where the union is over all solutions to (2.9) where $D = 2g + 2|G| - 2 - c(\vec{k})$. Therefore, if there are no solutions to (2.9), we have an empty union, so $\mathcal{F}_{D;\vec{k},E} = \emptyset$. Further, if there are solution to (2.9) but none of which that satisfy (2.5) then $\mathcal{F}_{D;\vec{k},E}$ would be a union of empty sets and thus empty itself. Therefore, from now on, we will always assume there is a solution to (2.9) and (2.5).

Let C_{δ_1} and C_{δ_2} be as defined in Corollary 7.2. Then

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_{\delta_1} + C_{\delta_2}} \frac{F_{\vec{k},E}(z)}{z^{D+1}} dz &= \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell+1} \sum_{\vec{t} \in \mathcal{R}'} \sum_{\nu \in \mathcal{M}} \xi_{r_n}^{-\vec{t} \cdot \vec{k}} E^{-\nu} \frac{1}{2\pi i} \oint_{C_{\delta_1} + C_{\delta_2}} \frac{A_{\vec{t},\nu}(z)}{z^{D+1}} dz \\ &= \left(\frac{1}{r_1 \cdots r_n} \right)^{\ell+1} \sum_{\vec{t} \in \mathcal{R}'} \sum_{\nu \in \mathcal{M}} \xi_{r_n}^{-\vec{t} \cdot \vec{k}} E^{-\nu} \sum_{i=1}^{\eta} P_{\vec{t},\nu,i}(D) q^{\frac{D}{c_i}} \\ &= \sum_{i=1}^{\eta} P_i(D) q^{\frac{D}{c_i}} \end{aligned}$$

where P_i is a quasi-polynomial of degree at most m_i .

Now, by Proposition 5.4, we know that

$$\frac{1}{2\pi i} \oint_{C_{\delta_1}} \frac{F_{\vec{k},E}(z)}{z^{D+1}} dz = -|\mathcal{F}_{D;\vec{k},E}|.$$

Moreover,

$$\left| \frac{1}{2\pi i} \oint_{C_{\delta_2}} \frac{F_{\vec{k},E}(z)}{z^{D+1}} dz \right| = O\left(q^{\left(\frac{1}{2} + \epsilon\right) \frac{D}{c_1}}\right)$$

where the implied constant is the maximum values of $F_{\vec{k},E}(z)$ on C_{δ_2} . □

Remark 7.5. If we let $c(\vec{\alpha})$ be any integers, then we could have that $\frac{D}{c_i} \leq \frac{D}{2c_1}$ and thus part of the main term could be absorbed into the error term. However, if we let $c(\vec{\alpha}) = |G| - \frac{|G|}{e(\vec{\alpha})}$, then we actually have that $\frac{D}{c_i} > \frac{D}{2c_1}$ for all $i = 1, \dots, \eta$. So for small enough ϵ , none of our main terms can be absorbed into the error term.

8. PROOFS OF THE MAIN THEOREMS

All that remains is to combine Proposition 7.4 and Lemma 4.2. From now on, we will fix the $c(\vec{\alpha}) = |G| - \frac{|G|}{e(\vec{\alpha})}$. But first, we present a little more notation in order to deal with the subgroups of G as used in Lemma 4.2.

As in Section 3, there is a natural bijection from $G \setminus \{e\}$ to \mathcal{R} . For every $H \subset G$, let \mathcal{R}_H be the image of H under this natural bijection. Recall that $\eta = \eta_G$ is the number of non-trivial divisors of $\exp(G) = r_n$. Then, for all $\vec{t} \in \mathcal{R}'$, $\nu \in \mathcal{M}$, $0 \leq a \leq r_n - 1$ and $1 \leq i \leq \eta_G$, define the analogous objects

$$\begin{aligned} \mathcal{R}_{H,\nu} &= \{\vec{\alpha} \in \mathcal{R}_H : \nu \vec{\alpha} = 0\} \\ \mathcal{R}_{H,\vec{t},\nu;a,i} &= \{\vec{\alpha} \in \mathcal{R}_{H,\nu} : c(\vec{\alpha}) = c_i \text{ and } \vec{t} \cdot \vec{\alpha} \equiv a \pmod{r_n}\} \\ m_{H,\vec{t},\nu;a,i} &= |\mathcal{R}_{H,\vec{t},\nu;a,i}| \\ m_{H,\vec{t},\nu,i} &= \max_{0 \leq a \leq r_n - 1} (m_{H,\vec{t},\nu;a,i}) \end{aligned}$$

$$m_{H,i} = \max_{\substack{\vec{t} \in \mathcal{R}' \\ \nu \in \mathcal{M}}} (m_{H,\vec{t},\nu,i})$$

Now,

$$m_{H,i} = m_{H,0,0,0,i} = |\{\vec{\alpha} \in \mathcal{R}_H : c(\vec{\alpha}) = c_i\}| = \phi_H(s_i)$$

since $c_i = |G| - \frac{|G|}{e(\vec{\alpha})}$ and $e(\vec{\alpha})$ is the order of $\vec{\alpha}$ as seen as an element in G . So, if $\vec{\alpha} \in \mathcal{R}_H$, then it can be seen as element in H and will have the same order. Notice, however, that we could have $\phi_H(s) = 0$ even if $\phi_G(s) \neq 0$.

Proof of Theorem 1.6. Lemma 4 and Corollary 3.3 tell us that

$$\mathcal{H}_{G,g}^*(\vec{k}, E) = \sum_{H \subset G} \sum_{\vec{d}(\vec{\alpha})} \mathcal{F}_{\vec{d}(\vec{\alpha}); \vec{k}, E}$$

where the inner sum is over all $\vec{d}(\vec{\alpha})$ that satisfy

$$(8.1) \quad \begin{aligned} d(\vec{\alpha}) &= 0, \vec{\alpha} \notin \mathcal{R}_H \\ d_j &= \sum_{\vec{\alpha} \in \mathcal{R}} \alpha_j d(\vec{\alpha}) \equiv k_j \pmod{r_j}, j = 1, \dots, n \\ \sum_{\vec{\alpha} \in \mathcal{R}} c(\vec{\alpha}) d(\vec{\alpha}) &= 2g + 2|G| - 2 - c(\vec{k}). \end{aligned}$$

Therefore, if there are no solutions to (??) and (2.5), then the above sum is empty and we have that $|\mathcal{H}_{G,g}(\vec{k}, E)| = |\mathcal{H}_{G,g}^*(\vec{k}, E)| = 0$. From now on, we will assume that there exists a solution to (??) and (2.5) so that the above sum is non-empty. Further, note that if $g \not\equiv 1 \pmod{|G|/|H|}$ for some H then there would be no solutions to (8.1) as this would correspond to a curve with a non-integer genus, which is impossible.

Moreover, if $H \cong \mathbb{Z}/s_1\mathbb{Z} \times \dots \times \mathbb{Z}/s_n\mathbb{Z}$ where $s_j | r_j$, then \mathcal{R}_H can be identified with the set

$$[0, \dots, s_1 - 1] \times \dots \times [0, \dots, s_n - 1] \setminus \{(0, \dots, 0)\}.$$

This allows us to apply Proposition 7.4 to obtain

$$|\mathcal{H}_{G,g}^*(\vec{k}, E)| = \sum_{H \subset G} \mu(G/H) \left(\sum_{j=1}^{\eta_H} P_{H,j;\vec{k},E}(2g) q^{\frac{2g+2|G|-2}{|G|-\frac{|G|}{s_{H,j}}}} + O\left(q^{\frac{(1+\epsilon)g}{|G|-\frac{|G|}{s_{H,1}}}}\right) \right)$$

where η_H is the number of non-trivial divisors of $\exp(H)$ and $1 = s_{H,0} < s_{H,1} < \dots < s_{H,\eta_H} = \exp(H)$ are the divisor of $\exp(H)$ and $P_{H,j;\vec{k},E}$ is a quasi-polynomial of degree at most $\phi_H(s_{H,j}) - 1$ if $g \equiv 1 \pmod{|G|/|H|}$ and identically the 0 polynomial otherwise. Since $\exp(H) | \exp(G)$ for all $H \subset G$ and $\phi_H(s_{H,j}) \leq \phi_G(s_{G,j})$, we can write

$$|\mathcal{H}_{G,g}(\vec{k}, E)| = \sum_{j=1}^{\eta} P_{j;\vec{k},E}(2g) q^{\frac{2g+2|G|-2}{c_j}} + O\left(q^{\frac{(1+\epsilon)g}{c_1}}\right)$$

where c_j and $\eta = \eta_G$ are as above and $P_{j;\vec{k},E}$ is a quasi-polynomial of degree at most $\phi_G(s_j) - 1$. □

Proof of Theorem 1.4. If we set $\ell = 0$, then we get $E = \emptyset$ is an empty matrix and thus the condition on it vanishes in $\mathcal{H}_{\vec{k}, \emptyset}(G, g)$. Therefore,

$$\begin{aligned} |\mathcal{H}(G, g)| &= \sum_{\vec{k} \in \mathcal{R}'} |\mathcal{H}_{\vec{k}, \emptyset}(G, g)| \\ &= \sum_{\vec{k} \in \mathcal{R}'} \sum_{j=1}^{\eta} P_{j; \vec{k}, \emptyset}(2g) q^{\frac{2g+2|G|-2}{c_j}} + O\left(q^{\frac{(1+\epsilon)g}{c_1}}\right) \\ &= \sum_{j=1}^{\eta} \sum_{\vec{k} \in \mathcal{R}'} P_{j; \vec{k}, \emptyset}(2g) q^{\frac{2g+2|G|-2}{c_j}} + O\left(q^{\frac{(1+\epsilon)g}{c_1}}\right) \\ &= \sum_{j=1}^{\eta} P_j(2g) q^{\frac{2g+2|G|-2}{c_j}} + O\left(q^{\frac{(1+\epsilon)g}{c_1}}\right) \end{aligned}$$

where P_j is a quasi-polynomial of degree at most $\phi_G(s_j) - 1$. To show that P_1 has exact degree, suppose

$$P_1(2g) = a_0(g)(2g)^{\phi_G(s_1)-1} + a_1(g)(2g)^{\phi_G(s_1-2)} + \dots$$

for some periodic function a_i with integer period. Now, our results, shows that

$$\sum_{j=0}^{c_1-1} q^{-\frac{j}{c_1}} |\mathcal{H}_{G, g+j}| \sim \sum_{j=0}^{N-1} a_0(g+j)(2g)^{\phi_G(s_1)-1} q^{\frac{2g+2|G|-2}{c_1}}.$$

Further, (1.1) then tells us that

$$\sum_{j=0}^{c_1-1} a_0(g+j) = C(K, G) \neq 0.$$

Therefore, a_0 will be non-zero for at least one integer in every interval of length c_1 . That is, a_0 is not identically 0 and P_1 has degree exactly $\phi_G(s_1) - 1$. □

9. $G = (\mathbb{Z}/Q\mathbb{Z})^n$

In this section we will determine the leading coefficient of $P_{\vec{k}, E, 1}$ and P_1 that appear in Corollaries 1.5 and 1.7 in the case that $G = (\mathbb{Z}/Q\mathbb{Z})^n$.

The reason we are able to determine the leading coefficient of P_1 in this case is that the genus and Möbius inversion formulas become simpler when $G = (\mathbb{Z}/Q\mathbb{Z})^n$. Indeed, in this case (2.1) becomes

(9.1)

$$2g + 2Q^n - 2 = \begin{cases} (Q^n - Q^{n-1}) \sum_{\vec{\alpha} \in \mathcal{R}} d(\vec{\alpha}) & d_j \equiv 0 \pmod{Q}, j = 1, \dots, n \\ (Q^n - Q^{n-1}) (\sum_{\vec{\alpha} \in \mathcal{R}} d(\vec{\alpha}) + 1) & \text{otherwise} \end{cases}$$

Therefore, by Theorem 1.6, we get that if $2g + 2Q^n - 2 \equiv 0 \pmod{Q^n - Q^{n-1}}$ then

$$\mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n, g}(\vec{k}, E) = \begin{cases} P_{\vec{0}, E}(2g) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}} & \vec{k} = \vec{0} \\ P_{\vec{k}, E}(2g) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}-1} & \vec{k} \neq \vec{0} \end{cases} + O\left(q^{(1+\epsilon)\frac{g}{Q^n-Q^{n-1}}}\right)$$

for some quasi-polynomial $P_{\vec{k},E}$ whose degree is at most $\phi_G(Q) - 1 = Q^n - 2$. In fact we will show the is, in fact, a polynomials and it has exact degree $Q^n - 2$. For the rest of this section we will always be assuming that $2g + 2Q^n - 2 \equiv 0 \pmod{Q^n - Q^{n-1}}$.

We see that in this case we get $c(\vec{\alpha}) = c(\vec{d}) = Q^n - Q^{n-1}$ for all $\vec{\alpha} \in \mathcal{R}$. Therefore, since we always assumed $c(\vec{\alpha})$ was arbitrary we can apply the results therein to this case $c(\vec{\alpha}) = c(\vec{d}) = 1$ and $D = \frac{2g+2Q^n-2}{Q^n-Q^{n-1}} \in \mathbb{N}$ in order to find the leading coefficient of $P_{\vec{k},E}$.

If we look back at where the quasi-polynomials come from, it is because we have a factor of the form $\zeta_{c_i}^{-kD}$ appearing in the constant $C_{\vec{t},\nu;a,i,k}$ in Lemma 7.1. Therefore, since in the case we can assume $c_i = 1$ for all i , these terms disappear and we in fact get that $P_{\vec{k},E}$ is a polynomial and not a quasi-polynomial.

Remark 9.1. By setting $D = \frac{2g+2Q^n-2}{Q^n-Q^{n-1}}$ instead of just $2g + 2Q^n - 2$, we are now counting by *conductor* instead of discriminant (genus). This is more analogous to what Bucur, et al. did in [1]. Because we can easily switch to counting by conductor is why it is easier to compute the constant in this case

In this setting, for all $\vec{t} \in \mathcal{R}'$ and $\nu \in \mathcal{M}$, we have that $A_{\vec{t},\nu}(z)$ will have poles of order $m_{\vec{t},\nu;a}$ when $z = (q\xi_Q^a)^{-1}$ where

$$m_{\vec{t},\nu;a} = |\{\vec{\alpha} \in \mathcal{R}_\nu : \vec{t} \cdot \vec{\alpha} \equiv a \pmod{Q}\}|.$$

Now, since $(\mathbb{Z}/Q\mathbb{Z})^n$ can be viewed as a vector space over the field $\mathbb{Z}/Q\mathbb{Z}$, we get that the action of ν and \vec{t} on \mathcal{R} are vector space morphisms. Therefore, the set

$$\{\vec{\alpha} \in \mathcal{R}_\nu : \vec{t} \cdot \vec{\alpha} \equiv a \pmod{Q}\} \subsetneq \mathcal{R}$$

unless $\nu = 0$, $\vec{t} = \vec{0}$ and $a = 0$. In which case we get

$$m_{\vec{0},0;0} = |\mathcal{R}| = Q^n - 1.$$

Therefore, combining the results of Section 7, we get that the leading coefficient of $P_{\vec{k},E}$ is

$$\begin{aligned} C_{\vec{k},E} &= \frac{1}{(Q^n - 2)!} (1 - q^{-1})^{Q^n - 1} \prod_P \left(\frac{|P|^{Q^n - 1} + (Q^n - 1)|P|^{Q^n - 2}}{(|P| + 1)^{Q^n - 1}} \right) \left(\frac{q}{Q^n(q + Q^n - 1)} \right)^\ell \\ &= \frac{1}{(Q^n - 2)!} \frac{L_{Q^n - 2}}{\zeta_q(2)^{Q^n - 1}} \left(\frac{q}{Q^n(q + Q^n - 1)} \right)^\ell \end{aligned}$$

where

$$(9.2) \quad L_m = \prod_{j=1}^m \prod_P \left(1 - \frac{j}{(|P| - 1)(|P| + j)} \right)$$

and

$$(9.3) \quad \zeta_q(s) = \sum_{F \in \mathbb{F}_q[X]} \frac{1}{|F|^s} = \frac{1}{1 - q^{1-s}}$$

is the zeta function where $|F| = q^{\deg F}$.

Notice that $C_{\vec{k},E}$ does not depend on \vec{k} or E . Therefore, if we set $\ell = 0$ and sum over all \vec{k} , we get

$$\mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n,g} = \sum_{\vec{k} \in \mathcal{R}'} \mathcal{H}_{(\mathbb{Z}/Q\mathbb{Z})^n,g}(\vec{k}, \emptyset)$$

$$\begin{aligned}
 &= P_{0,\emptyset} \left(\frac{2g+2Q^n-2}{Q^n-Q^{n-1}} \right) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}} + \sum_{\vec{k} \in \mathcal{R}} P_{\vec{k},\emptyset} \left(\frac{2g+2Q^n-2}{Q^n-Q^{n-1}} \right) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}-1} \\
 &\quad + O \left(q^{(1+\epsilon)\frac{g}{Q^n-Q^{n-1}}} \right) \\
 &= P \left(\frac{2g+2Q^n-2}{Q^n-Q^{n-1}} \right) q^{\frac{2g+2Q^n-2}{Q^n-Q^{n-1}}} + O \left(q^{(1+\epsilon)\frac{g}{Q^n-Q^{n-1}}} \right)
 \end{aligned}$$

where P is a polynomial of degree $Q^n - 2$ with leading coefficient

$$\begin{aligned}
 C &= C_{0,\emptyset} + \sum_{\vec{k} \in \mathcal{R}} C_{\vec{k},\emptyset} q^{-1} \\
 &= \frac{1}{(Q^n-2)!} \frac{q+Q^n-1}{q} \frac{L_{Q^n-2}}{\zeta_q(2)^{Q^n-1}}
 \end{aligned}$$

which is exactly the analogue of the constant in [1].

Since the condition $F_j(x_{q+1}) \neq 0$, where x_{q+1} is the point at infinity, is equivalent to saying $\deg(F_j) \equiv 0 \pmod{r_j}$ for $j = 1, \dots, n$, we get that if $\epsilon_{i,j} \in \mu_{r_j}$ for $i = 1, \dots, q+1$ and $j = 1, \dots, n$. Then as $g \rightarrow \infty$

$$\begin{aligned}
 &\frac{|\{C \in \mathcal{H}_{G,g} : \chi_{r_j}(F_j(x_i)) = \epsilon_{i,j}, i = 1, \dots, q+1, j = 1, \dots, n\}|}{|\mathcal{H}_{G,g}|} \\
 &= \frac{\frac{1}{Q^n} |\mathcal{H}_{G,g}(0, E)|}{|\mathcal{H}_{G,g}|} = \left(\frac{q}{Q^n(q+Q^n-1)} \right)^{q+1} \left(1 + O \left(\frac{1}{g} \right) \right)
 \end{aligned}$$

where the $\frac{1}{Q^n}$ factor in the first equality comes from the fact the leading coefficients of the F_j must satisfy $\chi_{r_j}(c_j) = \epsilon_{q+1,j}$.

Finally, from this result, the exact same argument will work to show that as $g \rightarrow \infty$,

$$\frac{|\{C \in \mathcal{H}_{G,g} : \#C(\mathbb{P}^1(\mathbb{F}_q)) = M\}|}{|\mathcal{H}_{G,g}|} = \text{Prob} \left(\sum_{i=1}^{q+1} X_i = M \right) \left(1 + O \left(\frac{1}{g} \right) \right)$$

where the X_i are *i.i.d.* random variables taking value 0, Q^n or Q^{n-1} such that

$$X_i = \begin{cases} Q^{n-1} & \text{with probability } \frac{Q^n-1}{Q^{n-1}(q+Q^n-1)} \\ Q^n & \text{with probability } \frac{q}{Q^n(q+Q^n-1)} \\ 0 & \text{with probability } \frac{(Q^n-1)(q+Q^n-Q)}{Q^n(q+Q^n-1)} \end{cases}.$$

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